Filter Banks - VI

Linear Phase Perfect Reconstruction QMF Banks

&

Cosine Modulated Filter Banks

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Perfect Reconstruction Filter Banks

In some applications, it is desirable to have a filter bank in which the analysis filters $H_k(z)$ are constrained to have linear phase. Such systems are called Linear Phase Filter Banks.

Aim: To design linear phase filters banks that satisfy Perfect Reconstruction (PR) property.

From the design examples (5.3.2 and 5.3.3 in the text), we can say that

- a perfect reconstruction system may not have a linear phase.
- a linear phase filter bank may not be a perfect reconstruction system

Necessary Condition:

In order to design FIR linear phase QMF banks which satisfy PR property, it is necessary to

- give up the Power Complimentary property.
- give up the relation $H_1(z) = H_0(-z)$

Power Complimentary Constraint must be Avoided.

Suppose H(z) and G(z) are linear phase filters. If these satisfy the power complimentary conditions, then H(z) and G(z) can be shown as sum of two delays viz., $H(z) = az^{-K} + bz^{-L}$ where K, L are integers. As a result, the responses of $H(e^{-jw})$ and $G(e^{-jw})$ are very restricted. Hence, we need to avoid Power Complimentary Condition on the analysis filters in order to obtain good responses.

Relation between $H_1(z) = H_0(-z)$ Must be Avoided

Condition for alias-free FIR QMF is:

$$H_1(z) = H_0(-z)$$

The distortion function is:

$$T(z) = (1/2)\,[\,H_1^2(z) - H_0^2(-z)\,]$$

If $H_0(z)$ has linear phase, then $T(z)$ has linear phase and phase distortion is eliminated. However, the amplitude distortion still persists, i.e. $T(e^{-jw})$ is not perfectly flat. However, it can be made very small by increasing the order of $H_0(z)$ (see example 5.2.1 in text) so that the system comes close to PR. But we can never achieve PR property exactly !

For perfect reconstruction, T(z) has to be a delay, that is H(z) has to be a sum of two delays, which is not useful. Hence it is necessary to give up the condition $H_1(z) = H_0(-z)$.

Design Examples:

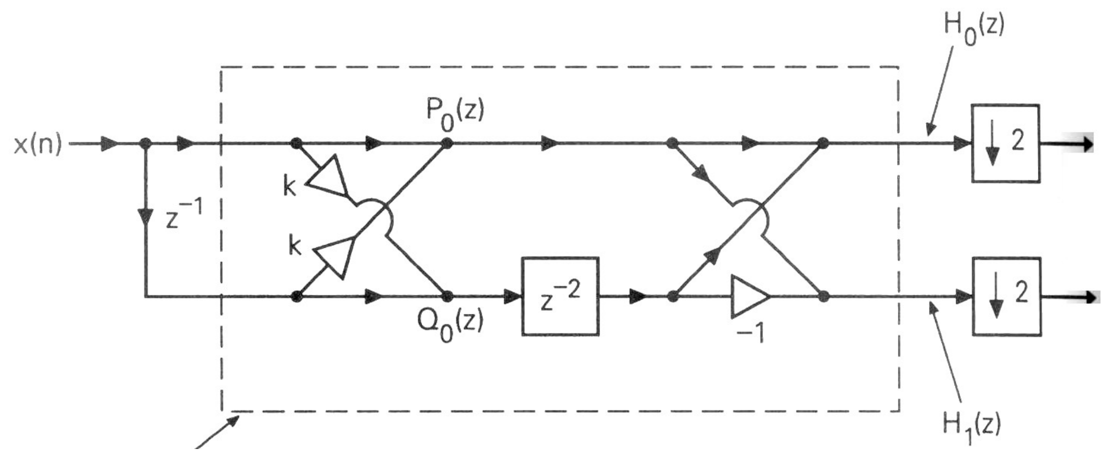
The condition for perfect reconstruction in FIR QMF banks which is really necessary and sufficient is:

$$H_0(z) = 1 \quad , \quad H_1(z) = z^{-1} \quad , \quad F_0(z) = z^{-1} \quad , \quad F_1(z) = 1$$

- This system is neither Power Complimentary nor satisfies the condition $H_1(z) = H_0(-z)$

Example 1.

Consider the following analysis bank:



Polyphase Matrix:

$$E(z) \;=\; \mathrm{T}\;.\;\Lambda(\mathrm{z})\;.\;\mathrm{T}_0$$

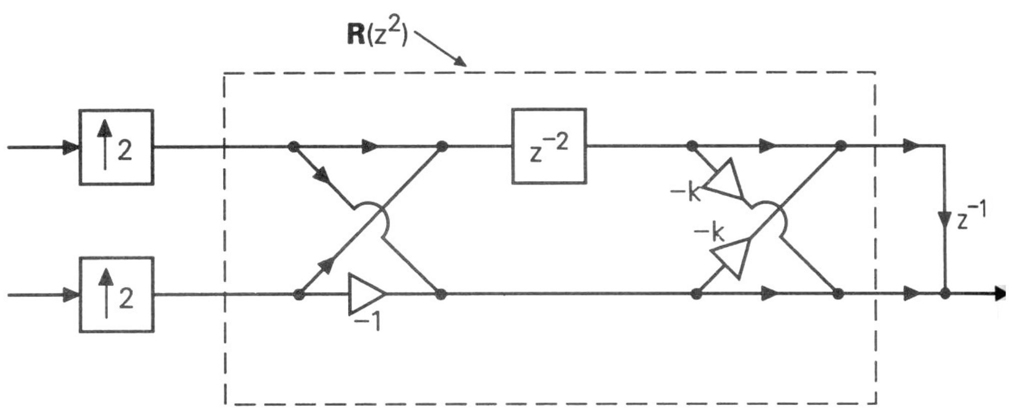
where

$$T_0 = \begin{bmatrix} 1 & \mathrm{k} \\ \mathrm{k} & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{z}^{-1} \end{bmatrix} \quad T_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Assume $\mathrm{k} \neq \pm 1$, so that T_0 is not singular.

Corresponding synthesis bank can be obtained as:
(for appropriate *c*)

$$R(\mathrm{z}) \;=\; \mathrm{c}z^{-1}\mathrm{E}^{-1}(z) \;=\mathrm{c}T_0^{-1}\begin{bmatrix} \mathrm{z}^{-1} & 0 \\ 0 & -1 \end{bmatrix}T_1^{-1}$$



The analysis and synthesis filters are verified to be

$$H_0(z) = 1 + kz^{-1} + kz^{-2} + kz^{-3},\ H_1(z) = 1 + kz^{-1} - kz^{-2} - kz^{-3}$$
$$F_0(z) = 1 - kz^{-1} - kz^{-2} + kz^{-3},\ F_1(z) = -1 + kz^{-1} - kz^{-2} + kz^{-3}$$

Note:

- Synthesis filters satisfy $F_0(z) = H_1(-z)$ and $F_1(z) = -H_0(-z)$ consistent with alias cancellation condition.
- Analysis filters evidently have linear phase, at the same time, are neither Power complimentary nor satisfy the condition $H_1(z) = H_0(-z)$.

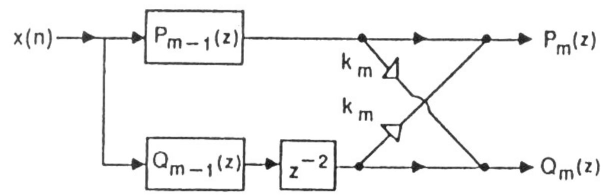
How is this perfect reconstruction system possible ?

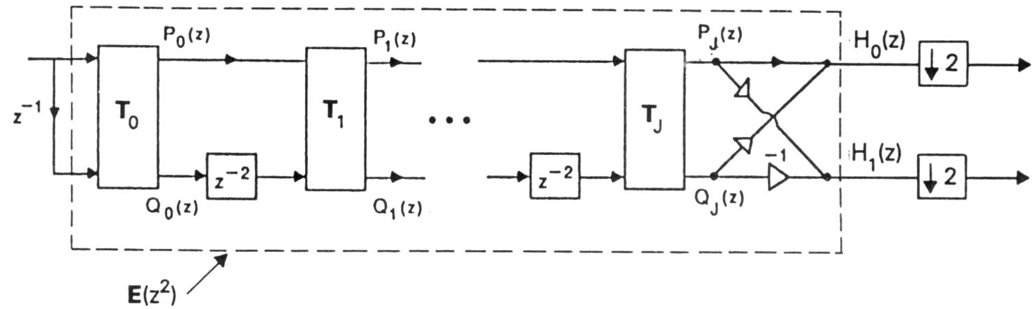
- Matrices $T_1(z)$ and $T_0(z)$ are carefully chosen.
- Choice of $T_0(z)$ is such that $Q_0(z)$ is the Hermitian image of $P_0(z)$ and choice of T_1 is such that $H_1(z)$ and $H_0(z)$ are the sum and difference of the image pair $P_0(z)$ and $z^{-2}Q_0(z)$ so that $h_0(n)$ is symmetric and $h_1(n)$ is anti-symmetric

Example 2.

(Example 7.2.2) The more general linear phase FIR PR QMF bank can be derived as shown in figure:

Analysis Bank:





$$P_{\text{m-1}}(\text{z}) = \sum_{\text{n=0}}^{\text{2m-1}} p_{m-1}(n) z^{-n} \quad , \quad Q_{\text{m-1}}(\text{z}) = \sum_{\text{n=0}}^{\text{2m-1}} q_{m-1}(n) z^{-n}$$

k_{m} is real

Let $Q_{m-1}(z)$ be the Hermitian image of $P_{m-1}(z)$ i.e. $Q_{m-1}(z) = z^{-(2m-1)} P_{m-1}(z)$ then,

$$P_m(\mathrm{z}) = P_{m-1}(z) + \mathrm{k_m z^{-2}} Q_{m-1}(z)$$

$$Q_m(\mathrm{z}) = \mathrm{k_m} P_{m-1}(z) + \mathrm{z^{-2}} Q_{m-1}(z)$$

Applying this to the analysis bank, we have

$$Q_J(\mathrm{z}) = \mathrm{z^{-N}} P_J(z^{-1})$$

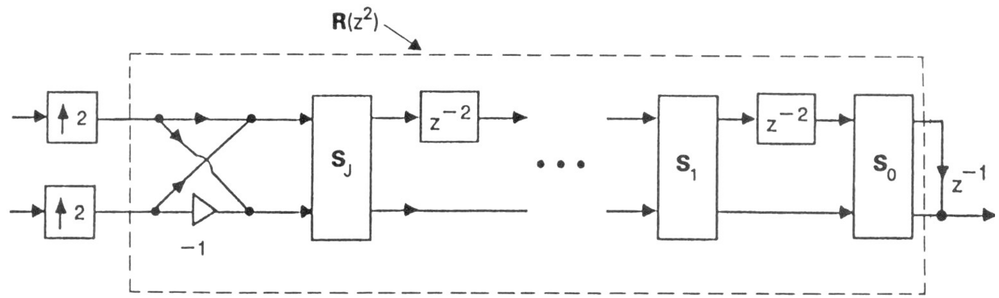
$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} P_J(z) \\ Q_J(z) \end{bmatrix}$$

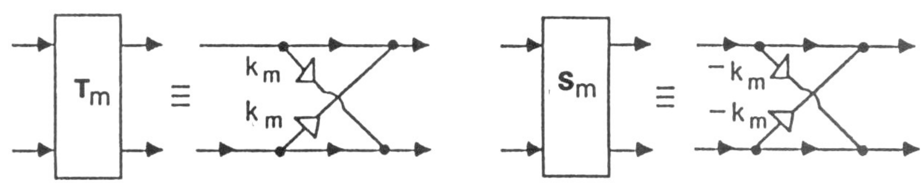
Hence, $\mathrm{H}_0(z) = z^{-\mathrm{N}} \mathrm{H}_0(z^{-1})$ and $\mathrm{H}_1(z) = - z^{-\mathrm{N}} \mathrm{H}_1(z^{-1})$ or

in terms of impulse response coefficients,

$$h_0(n) = h_0(N\text{-}n) \qquad \text{and} \qquad h_1(n) = - h_1(N\text{-}n)$$

Synthesis bank can be obtained by choosing $R(z) = cz^{-J}E^{-1}(z)$





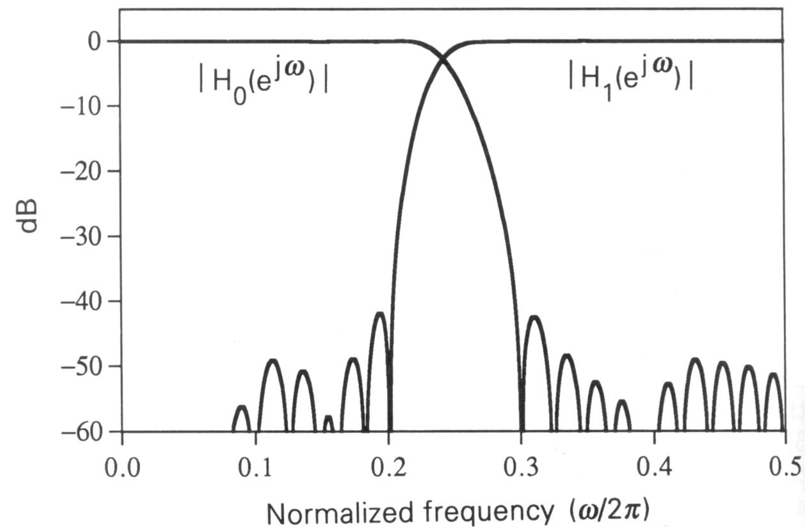
Example 3.

For J = 31, the lattice structure contains filters of order N = 63. The lattice coefficients should be optimized in order to minimize the objective function:

$$\varphi = \int_0^{\omega_p} \left[1 - \left|H_0(e^{j\omega})\right|\right]^2 d\omega + \int_{\omega_s}^{\pi} \left|H_0(e^{j\omega})\right|^2 d\omega$$

$$+ \int_{\omega_s}^{\pi} \left[1 - \left|H_1(e^{j\omega})\right|\right]^2 d\omega + \int_0^{\omega_p} \left|H_1(e^{j\omega})\right|^2 d\omega$$

The response of the optimized design is obtained as shown below. The transition bandwidth is 0.172π



Computational Complexity

The lattice structure has J+1 sections with two multipliers per section. However, each section can be rearranged, permitting an implementation with only one multiplier (and three adders) per section.

Therefore, the analysis bank requires:

MPU's = (1/4).(N + 1) + 1

APU's = (3/4).(N + 1) + 1 Where N is the order of the filter.

Cosine Modulated Filter Banks

Filter banks based on cosine modulation.

In these systems, all the M analysis filters are derived from a prototype filter $P_0(z)$ by cosine modulation.

Advantages:

- Cost of analysis bank is equal to that of one filter plus modulation overhead.

  Modulation can be done by fast techniques such as fast DCT.

  Synthesis filters have the same cost as analysis filters.
- Number of parameters to be optimized is very small because only the prototype has to be optimized.

Classification:

1. Approximate Reconstruction systems:- Ex.:- Pseudo QMF banks.
2. Perfect Reconstruction systems:- Ex.:- Cosine modulated PR filter banks.

1. Pseudo QMF banks:

- these are approximate reconstruction filter banks.

- Analysis and synthesis filters $H_k(z)$ and $F_k(z)$ are chosen so that only “adjacent-channel aliasing” is cancelled, and the distortion function $T(z)$ is only approximately a delay.

2. Cosine modulated PR systems:

- These are recently developed perfect reconstruction systems, basically are paraunitary systems.

- Analysis and synthesis filters $H_k(z)$ and $F_k(z)$ are chosen so that only “adjacent-channel aliasing” is cancelled, and the distortion function $T(z)$ is approximately only a delay.
- Retain all the simplicity and economy of pseudo QMF systems, yet have perfect reconstruction property.

Efficient Polyphase structure.

Consider the uniform DFT filter bank where exponential modulation is used. Let us now derive a class of filters with real coefficients by using cosine modulation with a slight modification i.e. replace M by 2M. This analysis bank, with 2M filters related as

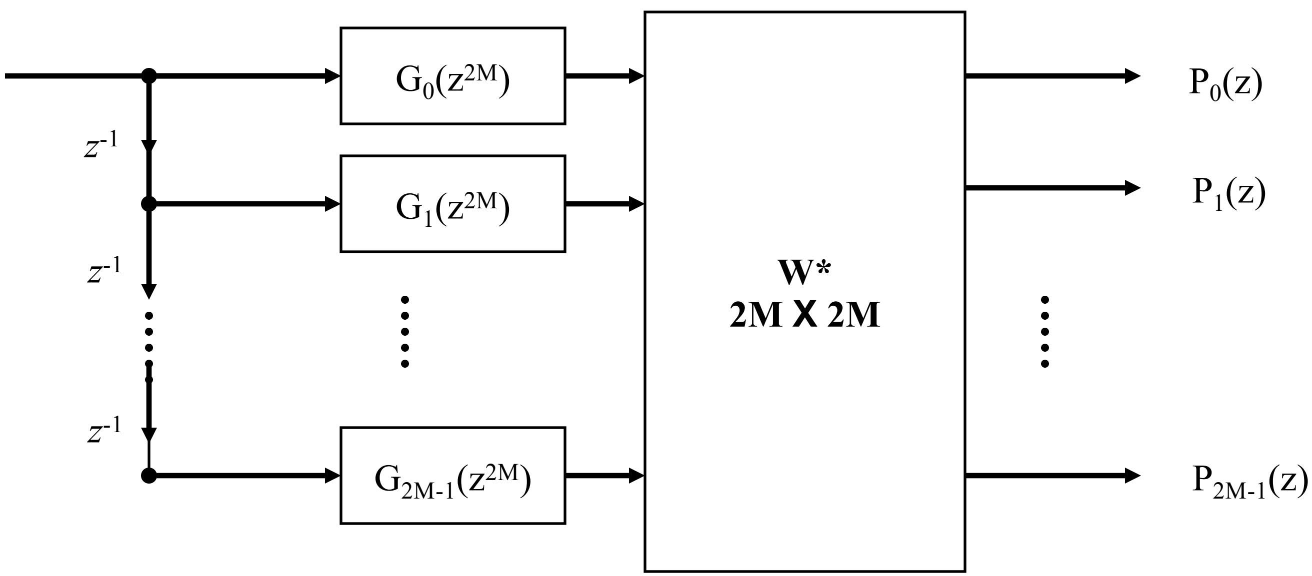
$$P_k(z) = P_0(zW^k) \quad \text{that is, } P_k(n) = P_0(n)\, W^{-kn}$$

(Let W here actually represents $W = W_{2M} = e^{-j2\pi/2M} = e^{-j\pi/M}$

W is the 2M X 2M DFT matrix)

$P_0(z)$: Prototype filter whose impulse response is real and symmetric w.r.t.w=0, typically low-pass with cut-off frequency $\pi/2M$

$G_k(z)$: Polyphase components of $P_0(z)$ for $0 \leq k \leq 2M-1$.



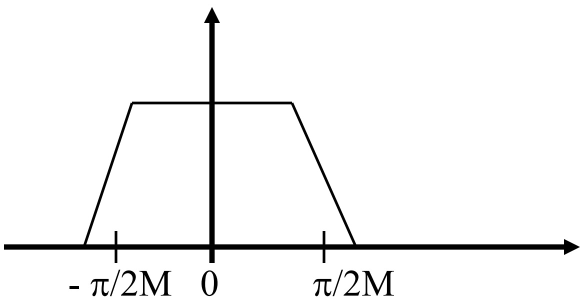


Fig. Magnitude response of $P_0(z)$

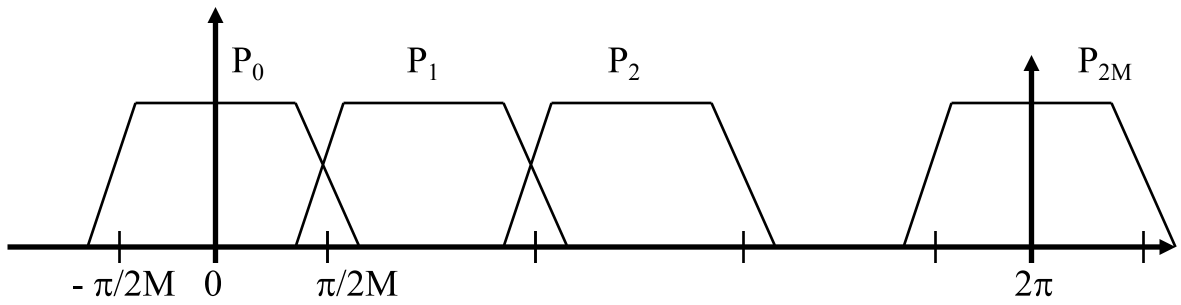


Fig. Magnitude response of shifted versions.

Typical pass-band width = π/2M

In order to make all filter band-widths equal after combining pairs, "*right shift*" the original set of 2M responses by an amount = π/2M i.e. replace z with $zW^{0.5}$
The complex filters $Q_k(z)$ are given in terms of $P_0(z)$ by:

$$Q_k(z) = P_0(zW^{k+0.5}), \qquad 0 \leq k \leq 2M-1$$

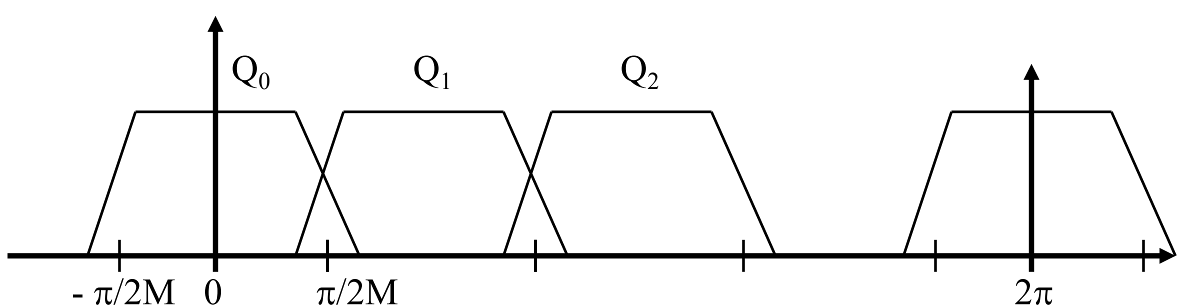
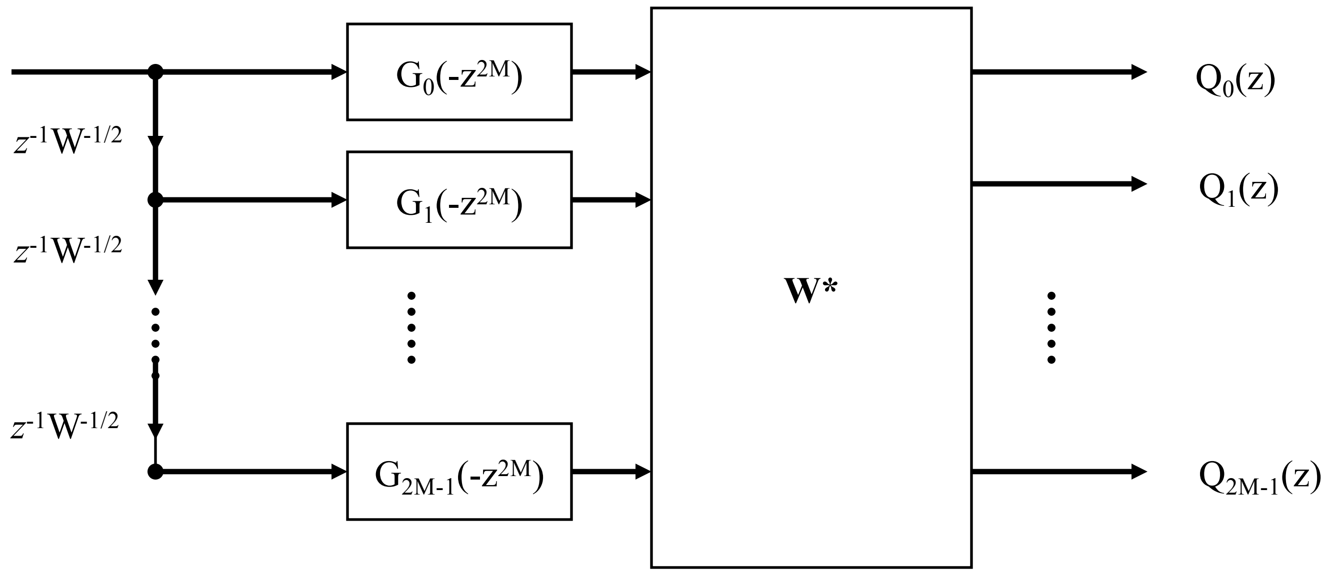


Fig. Shifting the responses by π/2M, by replacing z with $zW^{1/2}$



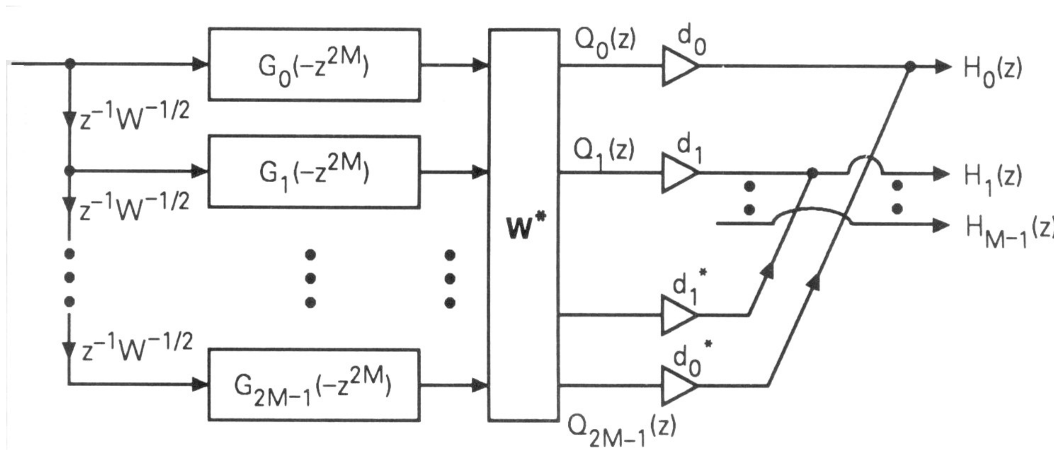
The magnitude responses of $Q_k(z)$ and $Q_{2M-1-k}(z)$ are images of each other w.r.t zero frequency, i.e.

$$|Q_k(e^{jw})| \quad = \quad |Q_{2M-1-k}(e^{-jw})|$$

Expression for analysis filters can now be written as:

$$H_k(z) \quad = \quad d_k\, Q_k(z) \; + d_k^* Q_{2M-1-k}(z)$$

where $d_k = e^{-j\theta_k} W^{(k+0.5)N/2}, \quad 0 \leq k \leq M-1$



We can implement the M analysis filters above as:

$$H_K(\mathrm{z}) = \sum_{\mathrm{n=0}}^{\mathrm{2M-1}} t_{kn} z^{\mathrm{-N}} G_n(-z^{2M}), \qquad 0 \le k \le M-1$$

where,

$$t_{kn} = W^{-(k+0.5)(n-\frac{N}{2})} . e^{j\theta_\mathrm{k}} + W^{(k+0.5)(n-\frac{N}{2})} . e^{-j\theta_\mathrm{k}}$$

$$= 2\cos\left(\frac{\pi}{M}(k+0.5)(n-\frac{N}{2}) + \theta_\mathrm{k}\right) \quad \text{where } \theta_\mathrm{k} = (-1)^k \pi/4$$

Thus the above Polyphase implementation figure for M-channel cosine modulated analysis bank can be redrawn as:

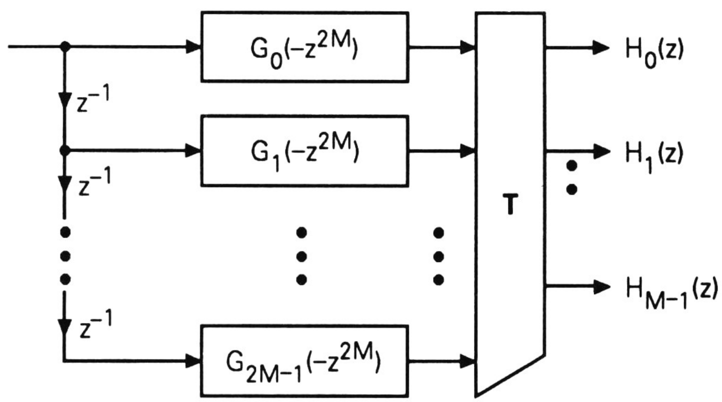


Fig. 1. Analysis filter bank

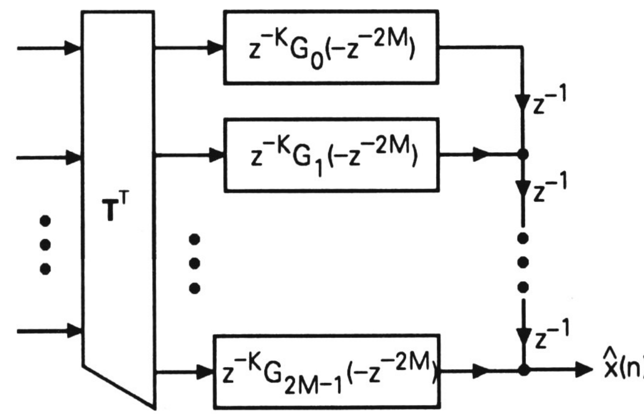


Fig.2 Corresponding Synthesis bank

Using noble identities, the most efficient Polyphase representation of cosine modulated filter bank is:

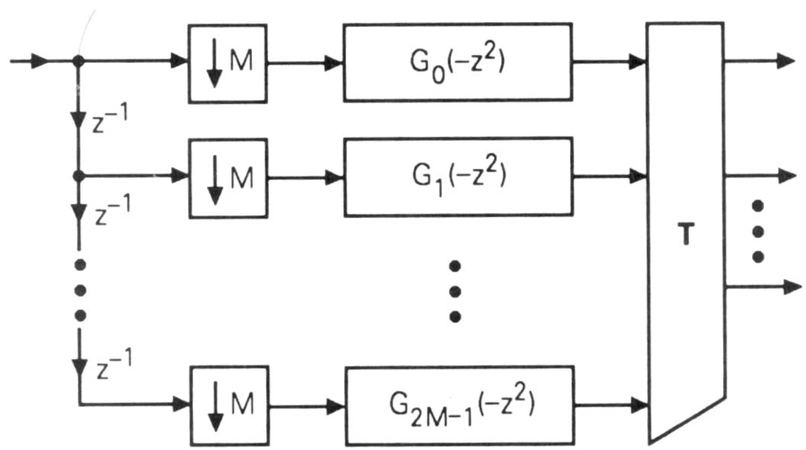


Fig.3 Analysis bank

DCT and DST matrices

Let matrix **C** be the discrete cosine transform (DCT) and **S** the discrete sine transform (DST). These are M x M matrices with elements

$$c_{kn} = \sqrt{\frac{2}{M}} \cos \frac{\pi}{M}(k+0.5)(n+0.5), \quad s_{kn} = \sqrt{\frac{2}{M}} \sin \frac{\pi}{M}(k+0.5)(n+0.5)$$

For special case of $N + 1 = 2mM$, the partition T in figure 1 in previous slides can be written as:

$$\mathbf{T} = [\ \mathbf{A_0} \quad \mathbf{A_1}\ ]$$

$$\mathrm{A}_0 = \sqrt{M}\Lambda_c(C - \Gamma S), \qquad \mathrm{A}_1 = -\sqrt{M}\Lambda_c(C + \Gamma S) \quad (m \text{ even})$$

$$\mathrm{A}_0 = \sqrt{M}\Lambda_c(C + \Gamma S), \qquad \mathrm{A}_1 = \sqrt{M}\Lambda_s(C - \Gamma S) \quad (m \text{ odd})$$

where Λ_c, Λ_s are MxM diagonal matrices with diagonal elements,

$$[\Lambda_c]_{kk} = \cos(\pi(k+0.5)m), \ \ [\Lambda_s]_{kk} = \sin(\pi(k+0.5)m).$$

Using the diagonal matrices Λ_c, Λ_s we can express

$$\mathrm{A}_0 = \sqrt{M}\left((\Lambda_c + \Gamma\Lambda_s)C - (-\Lambda_s + \Gamma\Lambda_c)S\right)$$

$$\mathrm{A}_1 = -\sqrt{M}\Gamma\left((\Lambda_c + \Gamma\Lambda_s)S + (-\Lambda_s + \Gamma\Lambda_c)C\right)$$

Cosine Modulated PERFECT RECONSTRUCTION Systems

- Among all FIR perfect reconstruction systems known today for arbitrary filter lengths, this system is perhaps the simplest(both in terms of design and implementation complexities)..

- It inherits all the simplicity and elegance of cosine modulated pseudo QMF system and yet offers Perfect Reconstruction property.

Expression for the Polyphase Matrix E(z)

The analysis bank structure as in figure 1 in previous slide has 2M polyphase components of prototype $P_0(z)$ in $G_k(z)$. Thus,

$$h(z) = T \begin{bmatrix} g_0(z^{2M}) & 0 \\ 0 & g_1(z^{2M}) \end{bmatrix} \begin{bmatrix} e(z) \\ z^{-M} e(z) \end{bmatrix}$$

$$= T \begin{bmatrix} g_0(z^{2M}) \\ z^{-M} g_1(z^{2M}) \end{bmatrix} e(z)$$

Where e(z) is the delay chain vector and $g_i(z)$ are diagonal matrices with

$$[g_0(z)]_{kk} = G_k(-z) \qquad [g_1(z)]_{kk} = G_{k+M}(-z)$$

Comparing with h(z) = E(z^M) e(z), we identify the polyphase matrix E(z) of the analysis bank as

$$E(z) = T\begin{bmatrix} g_0(z^2) \\ z^{-1}g_1(z^2) \end{bmatrix}$$

Using the partition $T = [\,A_0 \quad A_1\,]$, we have

$$E(z) = [A_0 \quad A_1]\begin{bmatrix} g_0(z^2) \\ z^{-1}g_1(z^2) \end{bmatrix}$$

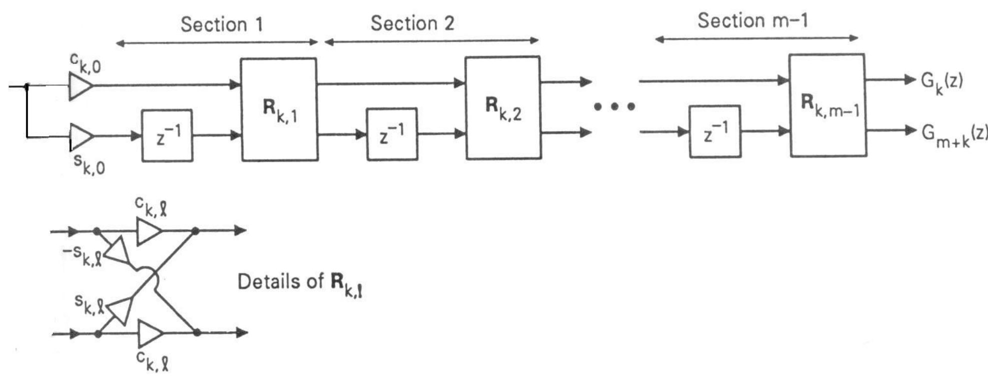
***Design Procedure*:**

Analysis filter responses are controlled by the prototype response $P_0(z)$.

It is sufficient to minimize only the stop-band energy ϕ_2 given by:

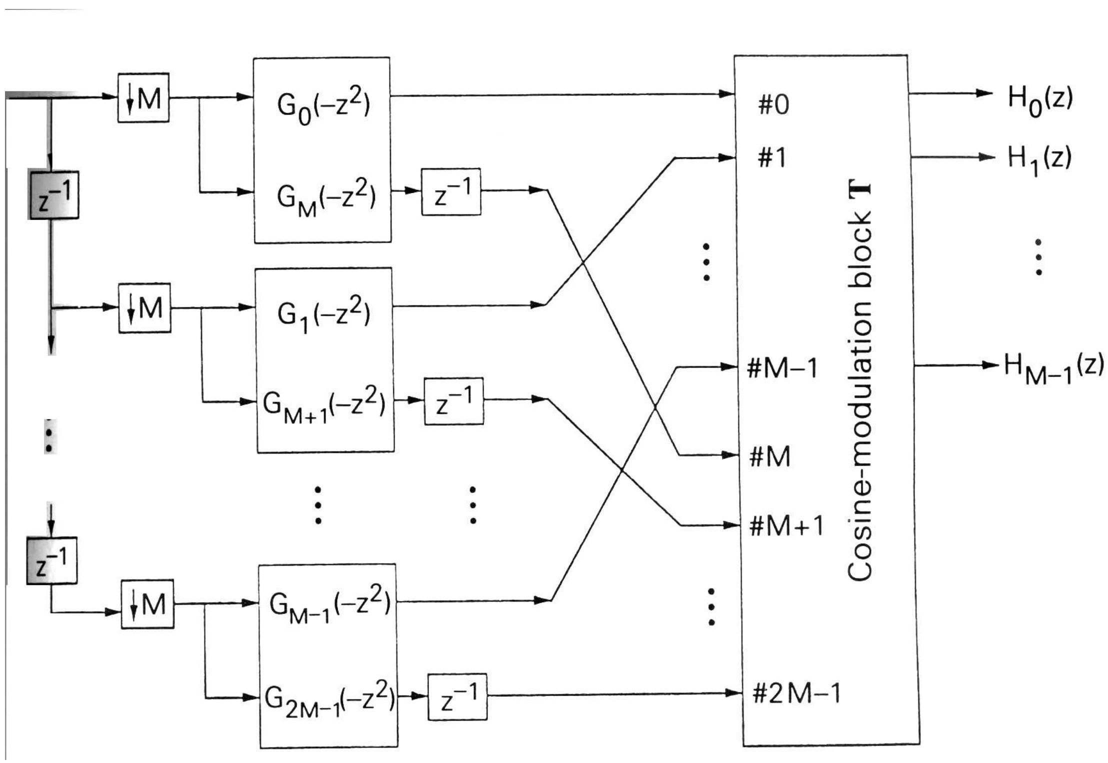
$$\varphi_2 = \int_{\frac{\pi}{2M}+\varepsilon}^{\pi} \left|P_0(e^{j\omega})\right|^2 d\omega$$

During optimization, it is necessary to impose the paraunitary constraint on $\boldsymbol{E}(z)$, equivalent to the condition that the FIR vector $\begin{bmatrix} G_k(z) \\ G_{M+k}(z) \end{bmatrix}$ be paraunitary. This vector can be implemented with the cascaded lattice structure as shown below:



In this, the transfer functions $G_k(z)$ and $G_{M+k}(z)$ are power complementary.

The cosine modulated analysis filter bank shown in figure 3 in previous slides will now become as shown in figure below. We now optimize the angles $\theta_{k,l}$ so as to minimize ϕ_2 .



Number of parameters to be optimized:

Since linear phase filters are implemented, only M/2 [(M-1)/2 for odd] lattice sections need to be optimized.

Advantages of the Cosine Modulated System

Summary:

1. Only the real coefficient prototype $P_0(z)$ is to be optimized in design, since all analysis filters are obtained from it, and due to paraunitary constraint on the polyphase matrix only half the number of parameters need to be optimized.

2. The objective function has to reflect only on the stop-band attenuation of $P_0(z)$.

3. Paraunitary constraint is automatically imposed during the design of prototype when we optimize the lattice coefficients

5. Implementation complexity is equal to the cost of the prototype $P_0(z)$ plus the modulation cost.

5. Modulation cost can be reduced by expressing analysis and synthesis banks in terms of fast DCT matrix.